On the Classification of Parametric Cubic Curves

Skip Thompson
Department of Mathematics & Statistics
Radford University
Radford, VA 24142
thompson@radford.edu

Abstract *We consider the question of characterizing the behavior of parametric curves whose components are cubic polynomials. When there is no chance of confusion, we will refer to such curves as cubic curves with the understanding that each of $x(t)$ and $y(t)$ are themselves cubic polynomials. We classify various types of parametric cubics using their defining coefficients. We show that this can be done in an elementary fashion accessible to students with a knowledge of Calculus. We discuss the manner in which the Maple Computer Algebra System can be used to explore this family of parametric curves.*

1 Introduction

The study of general planar algebraic cubic curves $F(x, y) = 0$ has long fascinated mathematicians. There is a vast and beautiful literature devoted to understanding planar cubics. It can be argued that the studies of these curves represented the beginnings of differential and algebraic geometry. Many good books, ranging from relatively elementary to very sophisticated treatments, are available on this subject. Very good introductions to planar cubics and certain higher degree curves include [1] and [4].

This paper considers the subfamily of planar cubics which have polynomial representations

$$(X(t), Y(t)) = (p_1(t), p_2(t)) \tag{1}$$

where

$$p_i(t) = \sum_{j=1}^{4} a_{ji}\, t^{4-j}.$$

In addition to being of intrinsic interest such curves are used extensively for planar interpolation and graphical design ([3], [5]). In this context piecewise cubics known as Bezier curves ([5]) have been studied extensively. One particularly nice way of studying general parametric cubics can be found in [3] which treats them as general algebraic curves, determines their properties, and then specializes the results to the family of general parametric cubics. We will use a less sophisticated approach in this paper in order to classify parametric cubics in an elementary manner that will be of interest to students and encourage further study of general cubics. We emphasize that we are not considering the entire family of planar cubics but are restricting our attention to the subfamily of cubics which have cubic polynomial parameterizations. By doing this we will be able to classify the curves in an elementary way and we will be able to use the Maple CAS ([2]) to visualize our classifications. Throughout the discussion we will identify and refer to parametric cubics as the points in the coefficient spaces defining the curves.

Parametric cubics of three types are of particular interest. A curve can have cusp, a loop point at which the curve intersects itself, or they can have neither a cusp nor a loop point. We will explore the conditions on the coefficients that must be satisfied by the coefficients of the cubic components for each of these types. Parametric cubics can also degenerate to other simpler curves.

Throughout this paper we assume that the leading nonzero coefficient of any polynomial is equal to 1 since scaling by the leading coefficient affects only the shape but not the curve type. Note also that the constant term does not affect the curve type since nonzero values amount to horizontal and vertical translations of the corresponding curve with $a_{4i} = 0$. We thus have a four parameter family of curves determined by a_{21}, a_{31}, a_{22}, and a_{32}, which we will denote by x, z, y, w, respectively, throughout this paper. For the purpose of notation, we will represent parametric curves using the mnemonic

$$C = \begin{bmatrix} a_{11} & a_{12} \\ x & y \\ z & w \end{bmatrix}.$$

2 Polynomial Parametric Cubics

Although some of the algebraic details of the calculations in this section are somewhat involved and are omitted, we note that Item 1 in §7 can be used to verify the details. To determine if curve (1) has a loop point, we need to find different values s and t that yield the same point on the parametric curve. To do this we need to solve the system

$$\begin{aligned} p_1(t) - p_1(s) &= 0 \\ p_2(t) - p_2(s) &= 0. \end{aligned}$$

Since $s = t$ is a solution of this polynomial system, we can remove the factor $t - s$ and obtain the deflated system

$$\begin{aligned} a_{11}s^2 + sx + sa_{11}t + z + tx + a_{11}t^2 &= 0 \\ a_{12}s^2 + sy + sa_{12}t + w + ty + a_{12}t^2 &= 0. \end{aligned}$$

The discriminants of the conics represented by this deflated system are $-3a_{11}^2$ and $-3a_{12}^2$. The conics are therefore ellipses if $a_{11}a_{12} \neq 0$. If $a_{1i} = 0$ the corresponding curve is a straight line (or a point). If we denote by α a root of the equation

$$\begin{aligned} &(a_{11}^2w^2 + a_{12}^2z^2 - 2a_{11}wa_{12}z - xa_{11}yw - xya_{12}z + a_{12}x^2w + za_{11}y^2) \\ &+ (a_{11}w - a_{12}z)Z + Z^2 = 0 \end{aligned}$$

the solution of the deflated system is given by

$$s = -\frac{a_{11}w - a_{12}z + \alpha}{a_{11}y - a_{12}x};\ t = \frac{\alpha}{a_{11}y - a_{12}x}.$$

The discriminant D of the Z quadratic determines the nature of s and t. After some manipulation, we find that

$$D = -3(a_{12}z - a_{11}w)^2 + 4\,(zy - xw)\,(a_{12}x - a_{11}y)\,.$$

If $D > 0$, s and t are real and unequal in which case the parametric curve has a loop point. If $D < 0$, s and t are complex so that the curve has neither a loop point nor a cusp. If $D = 0$, $s = t$ and the curve has a cusp or is degenerate.

A few words of explanation about degeneracies are necessary. The validity of our construction of s and t requires that $a_{11}y - a_{12}x \neq 0$. We define two matrices

$$C_X = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & x & x \\ 0 & 1 & 0 & z & 0 & z & 0 & z \end{bmatrix}$$

and

$$C_Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & y & y \\ 0 & 1 & 0 & w & 0 & w & 0 & w \end{bmatrix}.$$

Sixty-four basic curve types (including several redundancies) are obtained by using a column of C_X as the coefficients of $X(t)$ and a column of C_Y as the coefficients of $Y(t)$. It is straightforward to verify that if $a_{11}y - a_{12}x = 0$ and $D = 0$, the parametric cubic degenerates to one of several simpler curves including points, lines, half-lines, and parabolas. Also, certain special parametric cubics are obtained with $a_{11}y - a_{12}x = 0$ and $D < 0$. Our procedure for constructing s and t yields $s = t$ for these curves; there are neither distinct real nor complex values of s and t at which the curve intersects itself. The curves have neither a loop point nor a cusp. Although they somewhat resemble the curves for which $a_{11}y - a_{12}x \neq 0$ and s and t are complex conjugates, we choose to classify the former cubics as degenerate since they technically are different from the latter. Item 3 of §7 includes a detailed classification of all of the degenerate parametric cubics.

3 Some Special Cases

It is instructive to consider several special cases. For example, suppose each of $X(t)$ and $Y(t)$ is a (degenerate) curve without a quadratic term. We denote this case as

$$C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ z & w \end{bmatrix}.$$

In this case $D = -3(w - z)^2$. Consequently, if the coefficients of the linear terms are not equal, the parametric curve does not have a loop point or cusp since $D < 0$. If they are equal, the curve is a line.

Now consider the case in which $X(t)$ and $Y(t)$ are cubics without linear terms with the representation

$$C = \begin{bmatrix} 1 & 1 \\ x & y \\ 0 & 0 \end{bmatrix}.$$

In this case $D = 0$ and the curve has a cusp or is degenerate.

Another special case of interest is the one in which $X(t)$ is a quadratic without a linear term and $Y(t)$ is a cubic without a quadratic term with the representation

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & w \end{bmatrix}.$$

This case includes three prototypical cubics discussed in most books on the subject (e.g., [4]) to illustrate the basic behaviors of cubics. These curves are defined by

$$(X_1(t), Y_1(t)) = \left(t^2 - 1, t^3 - t\right) \tag{2}$$

$$(X_2(t), Y_2(t)) = \left(t^2 + 1, t^3 + t\right) \tag{3}$$

$$(X_3(t), Y_3(t)) = \left(t^2, t^3\right). \tag{4}$$

Their respective algebraic equations are given by

$$\begin{aligned} y^2 &= x^3 + x^2 \\ y^2 &= x^3 - x^2 \\ y^2 &= x^3. \end{aligned}$$

These curves are depicted in Figure 1. Curve (2) depicted in blue has a loop point, curve (4) depicted in black has a cusp, and curve (3) depicted in maroon has a neither a loop point nor a cusp. The discriminant of such a curve is equal to $-4w$ so that the type of the curve is determined by the coefficient of the linear term in $Y(t)$. If $w = 0$, the curve is similar to curve (4). If $w > 0$, the curve is similar to curve (2). If $w < 0$, the curve is similar to curve (3). An instructive exercise for students is to animate linear combinations of pairs of the curves to see how the behavior of the curves changes. Item 2 in §7 can be used for this purpose. If we extend this family to allow a nonzero linear term for $X(t)$, the discriminant becomes $-3z^2 - 4w$. In this case, the type of the curve is determined by the sign of this value.

To get a first glimpse at what the coordinate surfaces look like we consider the similar case in which $X(t)$ is cubic and $Y(t)$ is quadratic with the representation

$$C = \begin{bmatrix} 1 & 0 \\ x & 1 \\ z & w \end{bmatrix}.$$

In this case $D = -3w^2 + 4xw - 4z$. Setting $D = 0$ and solving for w we obtain the surfaces

$$w = \frac{2}{3}\left(x \pm \sqrt{x^2 - 3z}\right).$$

Note that if $z > \frac{1}{3}x$, these surfaces are not real and the corresponding curves do not have loop points or cusps. Checking the sign of D in the various regions carefully shows that for other points $D > 0$ if (x, z, w) is above the top w-surface or below the bottom w-surface. The corresponding curves have

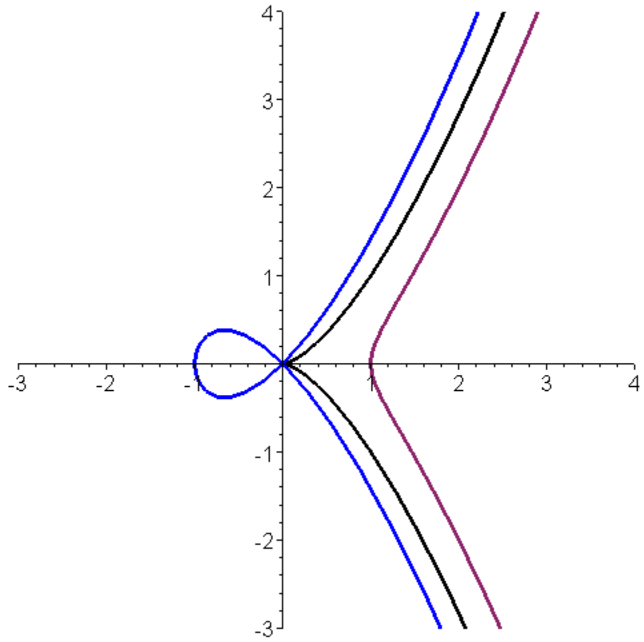

Figure 1: Prototypical Cubics

loop points. Finally, $D < 0$ if (x, z, w) is between the top and bottom w-surfaces. The corresponding curves have neither a loop point nor a cusp. Figure 2 depicts the set of points (x, y, z) for which $D = 0$. For clarity, the figure includes the cylinder $z = \frac{1}{3}x$ which abuts the w-surfaces.

Before considering the general cubic case, we consider one last special case (which we will again encounter in the general cubic case for $w = 0$). Such curves are represented by

$$C = \begin{bmatrix} 1 & 1 \\ x & y \\ z & 0 \end{bmatrix}.$$

Here, both $X(t)$ and $Y(t)$ are cubics but $Y(t)$ does not contain a linear term. In this case

$$D = -z(3z + 4xy - 4y^2).$$

Setting $D = 0$ leads to $z = 0$ or $z = \frac{4}{3}y(x - y)$. We see further that $D > 0$ if $z > 0$ and (x, y, z) is below the quadratic surface or if $z < 0$ and (x, y, z) is above the quadratic surface; and $D < 0$ if $z > 0$ and (x, y, z) is above the quadratic surface or if $z < 0$ and (x, y, z) is below the quadratic surface. Figure 3 depicts the quadratic surface along with the planes $z = 0$ and $x = y$.

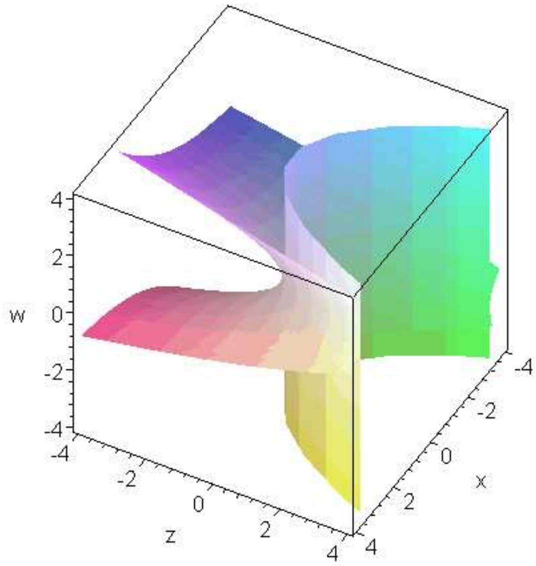

Figure 2: Level Surface for $y = 0$

4 General Cubic $x(t)$ and $y(t)$

In this case each of $X(t)$ and $Y(t)$ is a cubic without other restrictions on x, y, z, w and are represented by

$$C = \begin{bmatrix} 1 & 1 \\ x & y \\ z & w \end{bmatrix}.$$

In this case,

$$D = -3w^2 - 3z^2 + 6wz + 4xyw + 4xyz - 4x^2w - 4zy^2.$$

For $w = 0$ the set of points (x, y, z) for which $D = 0$ is the same as that for the last case in §3. For other values of w, we can solve for z to obtain

$$z = w + \frac{2}{3}\left(y(x-y) \pm \left(6xyw - 3wy^2 + x^2y^2 - 2xy^3 + y^4 - 3x^2w\right)^{1/2}\right).$$

Factoring the radicand we find that

$$\begin{aligned} z &= w + \frac{2}{3}y(x-y) \pm \frac{2}{3}\sqrt{(y^2 - 3w)(x-y)^2} \\ &= w + \frac{2}{3}y(x-y) \pm \frac{2}{3}|x-y|\sqrt{y^2 - 3w}. \end{aligned}$$

For $w \geq 0$ we must have $|y| \geq \sqrt{3w}$ in order for z to be real (unless $x = y$ in which case we obtain a degenerate curve). In addition to intersecting when $y = \pm\sqrt{3w}$, the two parts of the level surface, blue for the $+\surd$ and maroon for the $-\surd$ intersect along the portions of the line $[x, y = x, w]$ for

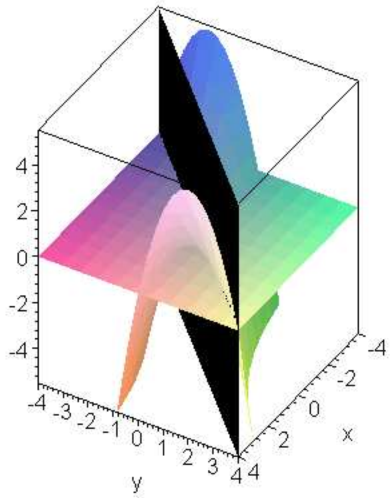

Figure 3: Level Surface for $w = 0$

which $|y| \geq \sqrt{3w}$. For $w > 0$ each of the two parts of the level surface separates into two pieces due to the restriction that $y > \sqrt{3w}$.

We next consider the sign of D. For all values of w the surfaces intersect only along segments of the line $[x, x, w]$. For $w = 0$ they also intersect in the plane $z = 0$. For $w > 0$ they also intersect along the lines $[x, \sqrt{3w}, w + \frac{2}{3}(x - \sqrt{3w})]$ and $[x, -\sqrt{3w}, w + \frac{2}{3}(x + \sqrt{3w})]$. Note that due to the symmetry of this function $z(x, y) = z(-y, -x)$ we can limit our attention to points for which y is nonnegative. For $w < 0$ it is straightforward to see that $D < 0$ if (x, y, z) is between the surfaces

$$z = w + \frac{2}{3}y(x - y) \pm \frac{2}{3}|x - y|\sqrt{y^2 - 3w}$$

and that $D > 0$ for (x, y, z) above both or below both of these surfaces. Now consider $w > 0$. In the middle region with $-\sqrt{3w} < y < \sqrt{3w}$ and $w > 0$ we have $D < 0$. This is the case since $y = 0$ yields $D = -3(w - z)^2 - 4x^2w < 0$ if $w \neq z$ or $x \neq 0$. The parametric cubics in this region contain neither a loop point nor a cusp. In fact, for $w > 0$ it is straighforward to show that $D > 0$ only for points in the "tent" protruding from the lower right of the surface with sides on the lines $[x, x, w]$ and $[x, \sqrt{3w}, w + \frac{2}{3}(x - \sqrt{3w})]$ (and, of course, its symmetrical image in the upper middle corner in Figure 4). These two tents yield the only curves with loop points for $w > 0$. They are determined by the conditions $x > y$, $y^2 > 3k$, and

$$w + \frac{2}{3}y(x - y) - \frac{2}{3}|x - y|\sqrt{y^2 - 3w} < z < w + \frac{2}{3}y(x - y) + \frac{2}{3}|x - y|\sqrt{y^2 - 3w}.$$

$D < 0$ for all other values (other than those on the plane $x = y$ and the line with $y = \sqrt{3k}$). Figure 4 contains an animated plot of a portion of these surfaces and the black plane $x - y = 0$ of degenerate curves obtained by varying w from -2 to 2.

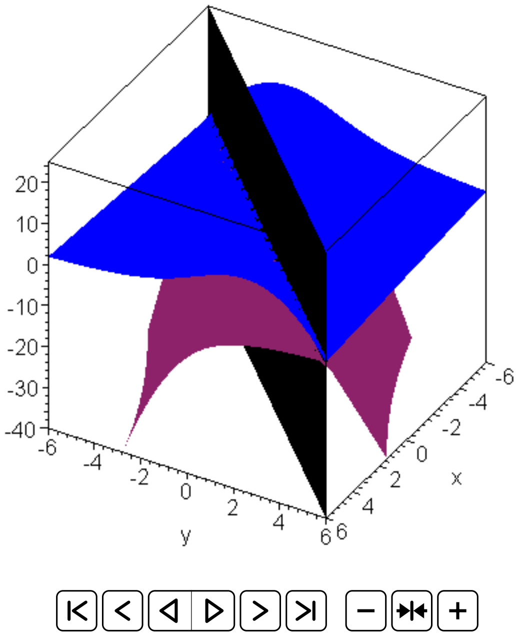

Figure 4. Animated Level Surfaces for $w \neq 0$

Recall that in the present case $a_{1,1} = a_{2,2} = 1$. The corresponding family of cubics thus contains no degenerate curves for which y is a quadratic or cubic function of x (or vice versa) since assuming otherwise leads to the condition that $a_{1,1} = 0$ or $a_{2,2} = 0$. Furthermore, the assumption that $y(t) = mx(t) + b$ is a linear function leads to the condition that $m = \dfrac{a_{j,1}}{a_{j,2}} = 1$ for $j = 1, 2, 3$. With our requirement that both $a_{1,1}$ and $a_{2,2}$ are 1, the only linear functions thus correspond to $x - y = 0$ (the intersection of the blue and maroon surfaces in the figures). More generally, linear functions are ones for which the coefficients of $y(t)$ are the same scalar multiple of the corresponding coefficients of $x(t)$.

5 Comments on Cusps and Curvature

We should point out that parametric cubics also can be characterized as in [9] by working with the curvature determined by the sign of

$$x'(t)y''(t) - y'(t)x''(t).$$

When interpreting the results from [9], the reader is cautioned that the discriminant used throughout the present paper is a negative constant multiple of the discriminant used in [9]. Item 1 of §7 performs equivalent calculations for any reader interested in exploring the curvature approach.

Here we mention a few interesting tidbits and outline some of the possibilities. With some work, we find that for points at which the curvature is 0, z satisfies $z = \dfrac{xw}{y}$. In this case $D = 0$ becomes

$$D = -3\left(\frac{w(x-y)}{y}\right)^2 = 0.$$

The inflection point surface $z = \dfrac{xw}{y}$ thus intersects the $D = 0$ surface when $w = 0$ or $x = y \neq 0$.

To determine the curves which have cusps, we solve the equations

$$X'(t) = 3u^2 + 2xu + z = 0$$

and

$$Y'(t) = 3u^2 + 2yu + w = 0.$$

We then equate the u solutions of these equations, and solve for z to obtain

$$z = -\frac{2\sqrt{3}x}{3}\sqrt{w} - w. \tag{5}$$

For a given value of w the intersection of this plane with the $D = 0$ level surface corresponds to the curves with cusps. Figure 4 depicts the level surfaces.

For $w > 0$ the curves of intersection of the $D = 0$ level surfaces with these planes are determined by the expressions

$$\frac{x^2\sqrt{3w} \pm \left(3x^4w + 36\sqrt{3}xw^{5/2} + 12\sqrt{3}x^3w^{3/2} + 54w^2x^2 + 27w^3\right)^{1/2}}{2x\sqrt{3w} + 3w}$$

with z given by (5).

6 Summary

In this paper we considered the various types of parametric cubic curves having polynomial components. We used the Maple CAS to perform several rather involved algebraic calculations and to visualize the surfaces on which the discriminant vanishes. It is anticipated that the discussion in this paper will encourage students to pursue more sophisticated treatments for general cubics and to develop an appreciation for these curves. Finally, we point out that both [3] and [9] contain quite interesting extensions of their results to piecewise parametric cubics represented as Bezier curves.

7 Supplemental Electronic Materials

1. Cubics.mws, a Maple worksheet for verifying the results and plots in this paper
2. ThreePrototypicalCurves.mws, a Maple worksheet for the three prototypical cubics in §3
3. SixtyFourCurves.mws, a Maple worksheet for degenerate curves

References

[1] Bix, R., *Conics and Cubics: A Concrete Introduction to Algebraic Curves*, Springer, 2006.

[2] Maplesoft, Waterloo Maple Inc., 615 Kumpf Avenue, Waterloo, Ontario, Canada, 2010.

[3] Patterson, R.R., Parametric Cubics as Algebraic Curves, *Computer Aided Geometric Design, Vol. 5,* pp. 139–159, 1988.

[4] Rutter, J.W., *Geometry of Curves*, Chapman & Hall, 2000.

[5] Sederberg, T.W., Degenerate Parametric Curves, *Computer Aided Geometric Design, Vol. 1*, pp. 301–307, 1984.

[6] Stone, M.C. and DeRose, T.D., A Geometric Characterization of Parametric Cubic Curves, *ACM Transactions on Graphics, Vol. 8*, pp. 147–163, 1989.

[7] Thompson, S., Maple Worksheet for Calculating Loop Points, 2011, `http://www.radford.edu/~thompson/Cubics/Cubics.mws`

[8] Thompson, S., Systematic Calculation of Loop Points for Parametric Curves, submitted.

[9] Wang, C.Y., Shape Classification of the Parametric Cubic Curve and Parametric B-spline Cubic Curve, *Comput. Aided Des., Vol. 113*, pp. 199–206, 1981.